MATH 5061 Lecture on 4/15/2020

[**Announcement**: PS5 due today, last PS6 posted.]

Plan for remaining lectures: harmonic forms, integral techniques, Bôchner formula.

Background: (Ref: Chern § 3.3, 3.4)

- Integration on manifolds

  (i) Measure theory : μ measure on $X \rightsquigarrow \int f\, d\mu$, $f: X \to \mathbb{R}$

  ★ (ii) "Oriented" Integration : e.g. $\iint_\Omega f(x,y)\,dx\,dy$ on $\Omega \subseteq \mathbb{R}^2$.

  $\rightsquigarrow$ Stokes' Thm (Green's , Divergence , Stokes' Thm)

$$``\int_\Omega \underbrace{d\omega}_{} = \int_{\partial\Omega} \underbrace{\omega}_{}"$$

orientation important

differential form

orientation: $dx \wedge dy = - dy \wedge dx$.

Def^n: A smooth manifold M^m is **orientable**

if $\exists$ atlas of coordinate charts s.t. all φ_{uv} are orientation-preserving diffeomorphisms between open sets of $\mathbb{R}^m$ (with std. orientation).

FACT: M^m orientable $\Leftrightarrow$ $\exists$ nowhere vanishing $\theta \in \Omega^m(M)$.

locally, $\theta = dx^1 \wedge \cdots \wedge dx^m \neq 0$ where $x^1, \ldots, x^m$ coord. fcn.

When (M^m, g) is Riemannian manifold, we define

" Volume form "

$$dV_g := \frac{\theta}{\|\theta\|_g} \overset{\text{loc.}}{=} \frac{dx^1 \wedge \cdots \wedge dx^m}{\|dx^1 \wedge \cdots \wedge dx^m\|_g} = \underbrace{\sqrt{g}}_{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^m .$$

Note: choice of orientation = choice of volume form

Recall: (V, g) $e_1, \dots, e_m$ O.N.B. $\rightsquigarrow$ $(V \wedge V, g)$ $\{e_i \wedge e_j\}$ O.N.B.

Def$^{\underline{n}}$: $\forall f \in C^\infty(M)$, M cpt

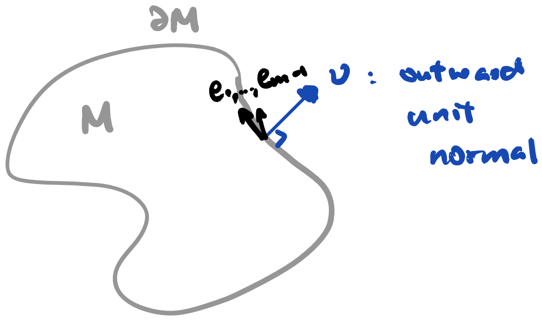
$$\int_M f := \int_M f \, dV_g \quad \leftarrow \text{integration of } m\text{-forms (define using local coord. } \int_U f(x^1, \dots, x^m) \sqrt{g} \, dx^1 \wedge \dots \wedge dx^m)$$

Stokes' Thm: Let M^m cpt, oriented manifold possibly with boundary ∂M.

$$\forall \eta \in \Omega^{m-1}(M), \quad \boxed{\int_M \underbrace{d\eta}_{m\text{-form}} = \int_{\partial M} \underbrace{\iota^* \eta}_{(m-1)\text{-form}}}$$

$\iota : \partial M \to M$ inclusion

Note: ∂M induces a "positive" orientation from M



$\{e_1, \dots, e_{m-1}\}$ pos. oriented basis for $T_p \partial M$

$\Updownarrow$

$\{\nu, e_1, \dots, e_{m-1}\}$ pos. oriented basis for $T_p M$

Remark: (1) Stokes' Thm holds w/o any metric g.

If (M^m, g) is Riemannian, $\iota_\nu (dV_g)$ is the volume form on ∂M.

(2) Divergence Thm is a special case of Stokes' Thm.

$$\begin{tikzcd} X \in \mathfrak{X}(M) \arrow[d, leftrightarrow, "g"', "\text{dual}"] \\ \omega \in \Omega^1(M) \end{tikzcd} \qquad \omega(Y) := \langle X, Y \rangle$$

Recall: $\operatorname{div} X := \sum_{i=1}^m \langle D_{e_i} X, e_i \rangle$ $\{e_i\}$ O.N.B.

FACT: $\underbrace{d(*\omega)}_{m\text{-form}} = (\operatorname{div} X) \, dV_g$ (Ex:)

(ie "div" = "$*d*$")

With this FACT,

$$\int_M (\operatorname{div} X)\, dV_g \overset{\substack{\text{FACT}\\ \downarrow}}{=} \int_M d(*\omega) \overset{\substack{\text{Stokes'}\\ \downarrow}}{=} \int_{\partial M} \iota^*(*\omega) \overset{\substack{(*)\\ \downarrow}}{=} \int_{\partial M} \langle X,\nu\rangle$$

Why $(*)$? At $p\in\partial M$, fix O.N.B. (pos. oriented)

$$\{\nu, \underbrace{e_1,\dots,e_{m-1}}_{T_p(\partial M)}\} \text{ on } T_pM \xleftrightarrow{\text{dual}} \{\nu^*, e_1^*,\dots,e_{m-1}^*\} \text{ on } T_p^*M.$$

Write $X_p = c\,\nu + \sum_{i=1}^{m-1} a_i e_i$

$\omega_p = c\,\nu^* + \sum_{i=1}^{m-1} a_i e_i^*$

$\Rightarrow \quad *\omega_p = c\, e_1^*\wedge\cdots\wedge e_{m-1}^* + \nu^*\wedge \boxed{/////////}$

$\Rightarrow \quad \iota^*(*\omega)_p = c\, e_1^*\wedge\cdots\wedge e_{m-1}^* = c\,(\text{Vol. form on } \partial M)$

and $c = \langle X,\nu\rangle$. $\square$

In particular, when M is "closed" (ie. cpt without boundary),

then $$\int_M \operatorname{div}(X) = 0 \qquad \forall\, X\in\mathfrak{X}(M).$$

Note: This is extremely useful, e.g. Bôchner methods.

Thm: Let (M^m, g) be a closed oriented manifold w/. $\mathrm{Ric}<0$.

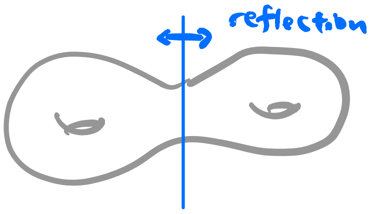
ie $\mathrm{Ric}(X,X)<0 \quad \forall\, 0\neq X\in T_pM$.

Then, the isometry group $\mathrm{Isom}(M) := \{\varphi: M\to M \text{ diffeo} \mid \varphi^* g = g\}$ is a finite group.

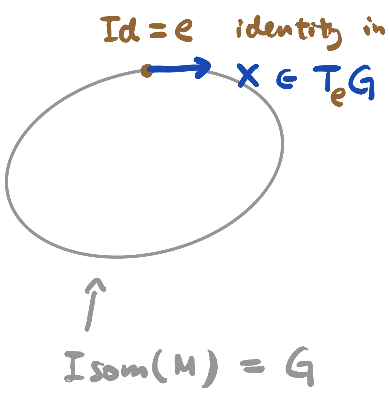
Remark: NOT true for positively curved spaces, e.g. (S^m, g_{round}) — $\mathrm{Isom} = SO(m+1)$ not finite.

Remark : (Σ^2, g) hyperbolic surface (ie $K \equiv -1$)

Can still have lots of isometries, just NOT a cts family of isometries.



Proof: Useful Fact : Isom(M,g) is a compact Lie group.



$$T_eG = \{ X \in \mathfrak{X}(M) \mid \overset{(\#)}{L_X g = 0} \}$$

$\Uparrow$

X : Killing vector fields

GOAL: $T_eG = \{0\}$

So, G would be a discrete group

cpt $\Rightarrow$ G finite.

Claim: Any Killing vector field on such manifold is trivial.

Fact about (#) : $L_X g = 0 \iff DX$ is skew-symm (1,1)-tensor

(Notation: $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$)   i.e. $(Y, Z) \longmapsto \langle D_Y X, Z \rangle$ is skew-symm.

Why? $$\underset{\substack{\Uparrow \\ \text{Symm.} \\ (0,2)\text{-tensor}}}{(L_X g)}(Y,Z) = X(\langle Y, Z\rangle) - \langle L_X Y, Z\rangle - \langle Y, L_X Z\rangle$$

$\downarrow$ D metric-compatible

$$= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle - \langle \underset{\substack{\| \\ D_XY - D_YX}}{[X,Y]}, Z\rangle - \langle Y, \underset{\substack{\| \\ D_XZ - D_ZX}}{[X,Z]}\rangle$$

D : torsion free

$$= \langle D_Y X, Z\rangle + \langle Y, D_Z X\rangle$$

So, $L_X g = 0 \iff \langle D_Y X, Z\rangle = -\langle D_Z X, Y\rangle \quad \forall Y, Z \in \mathfrak{X}(M)$

One direct consequence is $\boxed{\operatorname{div} X = 0 \qquad \forall \text{ Killing } X \in \mathfrak{X}(M)}$

$$\because \quad \operatorname{div} X \text{ "="} \operatorname{trace}(DX) = 0$$

Lemma: (Bŏchner formula for Killing vector fields)

Assume X is Killing vector field on (M^m, g).

$$\boxed{\frac{1}{2}\Delta \|X\|^2 = \|DX\|^2 - \operatorname{Ric}(X,X)} \quad \text{———} \quad (**)$$

Proof of (**) : **Recall**: In local O.N.B, $e_1, \dots, e_m$

$$\Delta f := \operatorname{tr}(\operatorname{Hess} f) = \sum_{i=1}^{m}\Big(e_i(e_i f) - (D_{e_i} e_i) f\Big)$$

First, $\frac{1}{2} e_i \|X\|^2 = \langle D_{e_i} X, X\rangle \overset{X \text{ Killing}}{=} -\langle D_X X, e_i\rangle$.

Next, $$\frac{1}{2}\Delta\|X\|^2 = \sum_{i=1}^{m}\Big(-e_i\langle D_X X, e_i\rangle + \langle D_X X, D_{e_i} e_i\rangle\Big)$$
$$= -\sum_{i=1}^{m}\langle D_{e_i} D_X X, e_i\rangle$$
$$= -\sum_{i=1}^{m}\Big(\langle R(e_i, X)X, e_i\rangle + \langle D_X D_{e_i} X, e_i\rangle + \langle D_{[e_i, X]} X, e_i\rangle\Big)$$

Notice : $$\langle D_X D_{e_i} X, e_i\rangle = X\big(\underbrace{\langle D_{e_i} X, e_i\rangle}_{\operatorname{div} X = 0}\big) - \langle D_{e_i} X, D_X e_i\rangle$$

$$= -\langle D_{e_i} X, D_{e_i} X\rangle + \langle D_{e_i} X, [e_i, X]\rangle$$

AND : $$\langle D_{[e_i, X]} X, e_i\rangle \overset{X \text{ Killing}}{=} -\langle D_{e_i} X, [e_i, X]\rangle$$

Putting it back, and sum over i.

$$\frac{1}{2}\Delta\|X\|^2 = -\operatorname{Ric}(X,X) + \|DX\|^2$$

$$\Rightarrow \quad 0 \underset{\because M \text{ closed}}{=} \int_M \frac{1}{2} \underset{\substack{\| \\ \operatorname{div}(\nabla(\ \ ))}}{\Delta} \|X\|^2 = \int_M \underbrace{\|DX\|^2}_{\geq 0} \underbrace{- \operatorname{Ric}(X,X)}_{\substack{\geq 0 \\ (\text{by } \operatorname{Ric} < 0)}} \Rightarrow X \equiv 0.$$

$\square$

Remark: If we only assume $\operatorname{Ric} \leq 0$, then any Killing field X is parallel.

Hodge theory for differential forms on (M^m, g)

Setup: (M^m, g) closed orientable, w/. volume form dV_g.

Recall: Hodge star operator : $* : \Omega^p(M) \longrightarrow \Omega^{m-p}(M)$

s.t. $\forall\, \alpha, \beta \in \Omega^p(M)$.

$$\boxed{\underbrace{\alpha}_{p} \wedge \underbrace{*\beta}_{m-p} = \langle \alpha, \beta \rangle \, dV_g}$$

Property: $*^2 = (-1)^{p(m-p)} \, id$ on $\Omega^p(M)$.

Recall: exterior derivative $d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ linear

on the inner product spaces where

$$(\Omega^p(M), \|\cdot\|_{L^2}) \qquad \langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha, \beta \rangle \, dV_g$$

$$= \int_M \alpha \wedge *\beta$$

We can define the "formal adjoint" of d w.r.t L^2- inner product above:

$$\delta : \Omega^{p+1}(M) \longrightarrow \Omega^p(M)$$

s.t. $$\langle \underbrace{\alpha, d\eta}_{(p+1)\text{-forms}} \rangle_{L^2} = \langle \underbrace{\delta\alpha, \eta}_{p\text{-forms}} \rangle_{L^2} \qquad \begin{array}{l} \forall\, \alpha \in \Omega^{p+1}(M) \\ \forall\, \eta \in \Omega^p(M) \end{array}$$

Def$^{\underline{n}}$: The **Hodge Laplacian** $\Delta : \Omega^p(M) \longrightarrow \Omega^p(M)$ is

$$\Delta \alpha := d(\delta \alpha) + \delta(d\alpha) \qquad \forall \alpha \in \Omega^p(M)$$

de Rham complex: $\dim M = m$.

$m/2$

$$\boxed{d^2 = 0}$$

$$\begin{tikzcd} 0 & \Omega^0(M) \arrow[l, "\delta"'] \arrow[r, shift left, "d"] & \Omega^1(M) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \cdots \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] & \Omega^p(M) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "d"] \arrow[loop above, "\Delta = d\delta + \delta d"] \arrow[d, phantom, "\Downarrow"] & \Omega^{p+1}(M) \arrow[l, shift left, "\delta"] \arrow[r, shift left] & \cdots \arrow[l, shift left, "\delta"] \arrow[r, shift left] & \Omega^m(M) \arrow[l, shift left, "\delta"] \\ & & & & \alpha & & & \end{tikzcd}$$

$$\boxed{\delta^2 = 0} \qquad \longleftrightarrow \ *$$

Remark: This generalizes the Laplace-Beltrami operator on $C^\infty(M) = \Omega^0(M)$.

(Ex: check this)

$$\because \quad \Delta^{\text{Hodge}} f = d\cancelto{0}{\delta(f)} + \delta d(f) = \delta(df) \overset{?}{=} \operatorname{div}(\nabla f)$$

Properties of Hodge-Laplacian

(i) Δ is "self-adjoint" (w.r.t. L^2), i.e. $\langle \Delta\alpha, \beta \rangle_{L^2} = \langle \alpha, \Delta\beta \rangle_{L^2}$

(ii) Δ is "non-negative definite"; ie. $\langle \Delta\alpha, \alpha \rangle_{L^2} \geq 0 \quad \forall \alpha \in \Omega^p(M)$

(iii) Δ commutes with $d, \delta, *$.

To prove these properties, we first prove:

Lemma: $\delta = (-1)^{mp+m+1} * d *$ on $\Omega^p(M)$

Cor: $\delta^2 = 0$. Pf: $\delta^2 = \pm (*d*)(*d*) = \pm(*d^2*) = 0$.

Pf of Lemma: Let $\alpha \in \Omega^p(M)$, $\eta \in \Omega^{p-1}(M)$.

Pointwise:
$$\begin{aligned} \langle \alpha, d\eta \rangle dV_g &= \alpha \wedge * d\eta = d\eta \wedge * \alpha \\ &= d(\eta \wedge * \alpha) - (-1)^{p-1} \eta \wedge d(*\alpha) \\ &= d(\eta \wedge * \alpha) + (-1)^{mp+m+1} \eta \wedge * (* d * \alpha) \end{aligned}$$

Integrate over M : $\langle \delta\alpha, \eta \rangle_{L^2} = \langle \alpha, d\eta \rangle_{L^2} = \overset{\text{M closed, Stokes'}}{0} + (-1)^{mp+m+1} \langle \eta, *d*\alpha \rangle_{L^2}$

$\blacksquare$

Proof of Properties of Δ :

(i) $\langle \Delta\alpha, \beta \rangle_{L^2} = \langle d\delta\alpha + \delta d\alpha, \beta \rangle_{L^2}$

$$= \langle d\delta\alpha, \beta \rangle_{L^2} + \langle \delta d\alpha, \beta \rangle_{L^2}$$

$$= \langle \delta\alpha, \delta\beta \rangle_{L^2} + \langle d\alpha, d\beta \rangle_{L^2} \quad (**)$$

$$= \langle \alpha, d\delta\beta \rangle_{L^2} + \langle \alpha, \delta d\beta \rangle_{L^2} = \langle \alpha, \Delta\beta \rangle_{L^2}$$

(ii) Take $\alpha = \beta$ in (i),

$$\langle \Delta\alpha, \alpha \rangle_{L^2} \overset{(**)}{=} \langle \delta\alpha, \delta\alpha \rangle_{L^2} + \langle d\alpha, d\alpha \rangle_{L^2}$$

$$= \|\delta\alpha\|_{L^2}^2 + \|d\alpha\|_{L^2}^2 \geq 0.$$

(iii) Exercise.

$\blacksquare$

Picture:

$$\begin{tikzcd} \Omega^{p-1}(M) \arrow[r, "d"] & \Omega^p(M) \arrow[loop above, "\Delta"] & \Omega^{p+1}(M) \arrow[l, "\delta"'] \end{tikzcd}$$

Hodge Decomposition Thm : $\exists$ L^2-orthogonal decomposition

$$\boxed{\overset{\infty\text{-dim}}{\Omega^p(M)} = \overset{\text{finite dim}}{\mathcal{H}_p} \oplus \overset{\infty\text{-dim}}{d(\Omega^{p-1}(M))} \oplus \overset{\infty\text{-dim}}{\delta(\Omega^{p+1}(M))}}$$

where $\mathcal{H}_p := \ker \Delta = \{ \alpha \in \Omega^p(M) \mid \Delta\alpha = 0 \}$

↳ space of "harmonic forms"

"FACT": $\mathcal{H}_p$ is finite dim'l and any L^2 harmonic form is smooth.

By (ii), $\Delta\alpha = 0 \iff d\alpha = 0$ & $\delta\alpha = 0$.

"Sketch of Proof": Consider $\Delta : \Omega^p(M) \longrightarrow \Omega^p(M)$

elliptic regularity
&
functional analysis

$$\Rightarrow \quad \Omega^p(M) = \ker(\Delta) \oplus \ker(\Delta)^{\perp}$$
$$= \mathcal{H}_p \oplus \mathrm{Im}(\Delta^*)$$
$$= \mathcal{H}_p \oplus \mathrm{Im}(\Delta)$$

So, $\forall\, \alpha \in \Omega^p(M)$, write $\alpha = \alpha_H + \Delta\beta = \alpha_H + d(\delta\beta) + \delta(d\beta)$
$$\in \mathcal{H}_p + d(\Omega^{p-1}) + \delta(\Omega^{p+1}).$$

It remains to check $\mathcal{H}_p$, $d(\Omega^{p-1})$ & $\delta(\Omega^p)$ are L^2-orthogonal.

Check: $d\Omega^{p-1} \perp_{L^2} \delta\Omega^{p+1}$?

$$\langle d\eta, \delta\theta \rangle_{L^2} = \langle d^2\eta, \theta \rangle_{L^2} = 0.$$ Similarly for others.

$\square$

Hodge Thm: $\exists!$ harmonic representation $\alpha_H \in \mathcal{H}_p$ in every de Rham cohomology class $[\alpha] \in H^p_{dR}(M) := \dfrac{\ker d}{\mathrm{im}\, d}$

Thus, $\mathcal{H}_p \cong H^p_{dR}(M)$

Proof: Let $\alpha_0 \in \Omega^p(M)$ closed, ie $d\alpha_0 = 0$.

$$H^p_{dR}(M) \ni \underset{\vee}{[\alpha_0]} := \{\alpha \in \Omega^p(M) \mid \alpha - \alpha_0 = d\eta \text{ for some } \eta \in \Omega^{p-1}(M)\}.$$

Goal: Find $\alpha_H \in \mathcal{H}_p$.

By Hodge decomposition, $\alpha_0 = \alpha_H + d\eta + \overset{=0}{\delta\theta}$

$$0 = d\alpha_0 = \underbrace{d\alpha_H}_{\overset{\shortparallel}{0}} + \underbrace{d^2\eta}_{\overset{\shortparallel}{0}} + d\delta\theta \quad \Rightarrow \quad d\delta\theta = 0.$$

But: $0 = \langle d\delta\theta, \theta \rangle_{L^2} = \langle \delta\theta, \delta\theta \rangle_{L^2} = \|\delta\theta\|^2_{L^2} \Rightarrow \delta\theta = 0.$

So, $[\alpha_0] = [\alpha_H] \in H^p_{dR}(M)$. This proves existence.

For uniqueness, suppose $[\alpha_0] = [\alpha_H] = [\alpha_H'] \in H^p_{dR}(M)$.

$$\Rightarrow \quad \alpha_H = \alpha_H' + d\eta \quad \text{for some } \eta \in \Omega^{p-1}(M).$$

$$\Rightarrow \quad \underbrace{\delta \alpha_H}_{\substack{\| \\ 0}} = \underbrace{\delta \alpha_H'}_{\substack{\| \\ 0}} + \delta d\eta$$

$$\Rightarrow \quad \delta d\eta = 0$$

$$\Rightarrow \quad d\eta = 0$$

So, $\alpha_H = \alpha_H'$. This proves uniqueness.

Remark: $d\alpha_0 = 0$, want to solve for η s.t.

$$\delta(\alpha_0 + d\eta) = 0$$

i.e. $$\delta d\eta = -\delta\alpha_0$$